Safety Comparison of Centralized and Distributed Aircraft Separation Assurance Concepts

John Shortle, member
Yimin Zhang
Systems Engineering & Operations Research, George Mason University, Fairfax, VA 22030

***Abstract*—This paper presents several models to compare centralized and distributed automated separation assurance concepts in aviation. In a centralized system, safety-related functions are implemented by common equipment on the ground. In a distributed system, safety-related functions are implemented by equipment on each aircraft. Failures of the safety-related functions can increase the risk of near mid-air collisions. Intuitively, failures on the ground are worse than failures in the air because the ground failures simultaneously affect multiple aircraft. This paper evaluates the degree to which this is true. Using region-wide models to account for dependencies between aircraft pairs, we derive the region-wide expectation and variance of the number of separation losses for both centralized and distributed concepts. This is done first for a basic scenario involving a single component/function. We show that the variance of the number of separation losses is always higher for the centralized system, holding the expectations equal. However, numerical examples show that the difference is negligible when the events of interest are rare. Results are extended to a hybrid centralized-distributed scenario involving multiple components/functions on the ground and in the air. In this case, the variance of the centralized system may actually be less than that of the distributed system. The overall implication is that the common-cause failure of the ground function does not seriously weaken the overall case for using a centralized concept versus a distributed concept.**

***Index Terms*—aviation safety, separation assurance, near mid-air collisions**

I. Introduction

To increase capacity in the national air transportation system, future aviation concepts will rely on increased levels of automation to separate aircraft. Two examples of automated separation-assurance concepts are the Advanced Airspace Concept (AAC) [1], [2], [3], [4], [5] and Autonomous Flight Rules (AFR) [6]. Both concepts use increased levels of automation to detect conflicts (potential losses of separation between aircraft) and to provide and execute resolution maneuvers.

A fundamental difference between the two concepts is that AAC is a *centralized* concept, while AFR is a *distributed* concept. Figure 1 shows an abstract representation of this difference. In the centralized system, the conflict detection and resolution (CD&R) function is implemented by common equipment on the ground, represented by a box in the figure. In the distributed system, the CD&R function is implemented by equipment aboard each aircraft. The dashed lines represent communication channels for surveillance information and trajectory resolutions.

The objective of this paper is to do a comparative safety analysis of a centralized system versus a distributed system with respect to failures of the generic safety-related functions. This paper also considers *hybrid* centralized-distributed systems in Section V (Figure 4). As a disclaimer, this is not a direct analysis of the AAC and AFR concepts. It is rather an abstracted analysis of generic centralized and distributed systems. Many complexities that might otherwise exist between such systems are ignored – for example, differences between global and local decision making, differences in situational awareness, differences in times to communicate resolutions, and so forth.

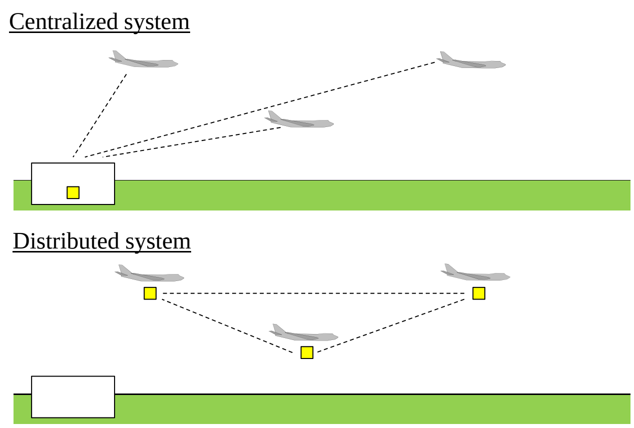

Fig. 1. Comparison of a centralized and distributed system; safety-related functions are represented by boxes.

The approach is to analytically evaluate the impact of safety-function failures (boxes in Figure 1) on the total numbers of separation losses. Failures of the safety-related functions do not *cause* separation losses; they simply remove the safety net that would otherwise exist to prevent them. Intuitively, a failure of the function on the ground in the centralized system is worse than a failure of the function in the air in the distributed system, since a ground failure simultaneously affects a large number of aircraft. This paper evaluates the degree to which this is true.

A key aspect of the analysis is the inherent dependence of separation losses between distinct aircraft pairs. In the centralized system, separation losses are dependent because of the common-cause failure of the ground function. If the ground function fails, the probability of a separation loss increases for *all* aircraft in some volume of airspace. Separation losses are also dependent in the distributed system, though to a lesser degree. If the air-based function on aircraft i fails, the probability of a separation loss increases for all aircraft pairs ij involving the index i.

This type of dependence is not captured in a pairwise safety model (e.g., [7], [8], [9], [10], [11], [12]). In a pairwise model, safety metrics are evaluated by considering two flights in isolation. For example, let $I_{ij} = 1$ if there is a near mid-air collision (NMAC) between aircraft i and aircraft j; let $I_{ij} = 0$ otherwise. The objective is to estimate $\mathrm{E}[I_{ij}] = \Pr\{I_{ij} = 1\}$. From a pairwise safety measure, it is possible to extrapolate to a region-wide measure by summing over the set of candidate aircraft pairs. For example, let

$$T \equiv \text{total \# of near mid-air collisions} = \sum_{i<j} I_{ij}.$$

(The range of summation, $i < j$, implies that an NMAC between aircraft i and j is treated as one event, not two.) The expected total number of NMACs is:

$$\mathrm{E}[T] = \mathrm{E}\left[\sum_{i<j} I_{ij}\right] = \sum_{i<j} \mathrm{E}[I_{ij}]. \tag{1}$$

The expectation can be brought inside the sum regardless of whether or not I_{ij} are independent. If we suppose that $\mathrm{E}[I_{ij}] = p$ for all $i \neq j$, then (1) becomes

$$\mathrm{E}[T] = \sum_{i<j} \mathrm{E}[I_{ij}] = \left(\frac{n^2 - n}{2}\right) p,$$

where n is the number of aircraft. That is, the region-wide result is the pairwise result p multiplied by the number of aircraft pairs.

However, this type of extrapolation from a pairwise model to a region-wide model cannot be done for other probabilistic metrics such as $\mathrm{var}[T]$ or $\Pr\{T \geq 2\}$. Such metrics are useful to reflect the possibility of multiple simultaneous events – for example, the probability that a failure of the ground function leads to multiple NMACs. In particular, the dependence of I_{ij} prevents the variance from being brought inside the summation: $\mathrm{var}[T] \neq \sum_{i<j} \mathrm{var}[I_{ij}]$. Thus, calculation of $\mathrm{var}[T]$ cannot be directly obtained from a pairwise model.

The main contributions of this paper are the following:

- We derive analytical expressions for the expectation and variance of the number of separation violations ($\mathrm{E}[T]$ and $\mathrm{var}[T]$) for both centralized and distributed systems, based on development of region-wide safety / reliability models of the two systems.
- We prove that if the underlying models are sufficiently simple, $\mathrm{var}[T]$ is always larger for the centralized system, holding $\mathrm{E}[T]$ equal between the two systems. This suggests a greater likelihood for multiple simultaneous separation losses for the centralized system. However, for more complex models, it is possible to construct counter examples where the variance is larger for the distributed system.
- We demonstrate through numerical examples that – even when the variance is theoretically higher for the centralized system – practically speaking, this difference is negligible when the events of interest are rare.

The overall implication of these results is that the common-cause failure of the ground function does not seriously weaken the overall case for a centralized concept versus a distributed concept.

Other related papers on aircraft reliability and safety include the following: [13] identifies optimal inspection policies for aircraft components whose failures may be hidden. [14] gives a method for determining the return on investment for health management approaches; a case study is given for a multi-functional display on a Boeing 737. [15] gives a model for survival of a system with an abort policy; an example is a multi-engine aircraft that can abort its mission after one engine fails. [16] gives a method for estimating the remaining life of safety-critical components in domains such as nuclear power and aviation. These papers do not consider issues associated with near mid-air collisions, which is inherently a multi-aircraft problem.

II. Models for Single Safety-Related Function

This section defines models for a centralized system and a distributed system with respect to failures of a single function for conflict detection and resolution (as in Figure 1). More complex models will be given in Section V. The centralized and distributed systems are assumed to be identical except for the implementation of one function. In the centralized system, the function is implemented by equipment on the ground.[1] In the distributed system, the function is implemented by equipment on each aircraft.

Centralized system: Let A_c denote the random state of the ground function, where $A_c = 0$ denotes a failed state and $A_c = 1$ denotes a working state. Let $r \equiv \Pr\{A_c = 0\}$ denote the failure probability of the ground function. For each distinct aircraft pair, the probability of a separation loss depends on A_c. If $A_c = 0$, the separation-loss probability is p_0. If $A_c = 1$, the probability is p_1, where $p_1 \leq p_0$. Given A_c, separation losses between distinct aircraft pairs are independent. That is, conditional on A_c, the total number of separation losses is a binomial random variable with probability parameter p_{Ac}.

Distributed system: Let A_d^i denote the random state of the function on aircraft i, where $A_d^i = 0$ denotes a failed state and $A_d^i = 1$ denotes a working state. Let $q \equiv \Pr\{A_i^d = 0\}$ denote the failure probability of the function on each aircraft. The states A_d^i are assumed to be independent of each other. For each distinct aircraft pair (i, j), the probability of a separation loss depends on A_d^i and A_d^j. If $A_d^i = 0$ and $A_d^j = 0$ (both functions are failed), the separation-loss probability is p_{00}. If $A_d^i = 1$ and $A_d^j = 1$ (both functions are working), the probability is p_{11}. If $A_d^i = 1$ and $A_d^j = 0$ (or vice-versa), the probability is $p_{01} \equiv p_{10}$. It is assumed that $p_{11} \leq p_{01} \leq p_{00}$. Conditional on A_d^i, separation losses between distinct aircraft pairs are independent.

[1]Throughout this paper, *on the ground* refers generically to something that is *not on the aircraft* and impacts all aircraft in the region. For example, a satellite system, whose failure would affect all aircraft, would be generically referred to as a ground system.

Remark 1: In these models, *separation loss* refers generically to a close proximity event that must be specifically defined via, say, lateral and vertical separation minima. For example, a separation loss could be defined as one of the following events (e.g., [17]):

- Standard loss of separation (LOS): ≤ 5 nautical mile (nm) lateral separation, and $\leq$ 1,000 ft vertical separation,
- Near mid-air collision (NMAC): $\leq$ 500 ft lateral, $\leq$ 100 ft vertical,
- Collision: $\leq$ 100 ft lateral, $\leq$ 30 ft vertical.

The numerical values for p_0, p_1, and p_{01} depend on the definition of separation loss used in the model. Lower-consequence events, such as 5-nm separation losses, would correspond to higher separation-loss probabilities, while higher-consequence events, such as NMACs, would correspond to lower probabilities.

Remark 2: The independence assumption is reasonable if separation loss refers to a standard LOS or an NMAC, but not if it refers to a collision. Let I_{ij} denote the indicator of separation loss between aircraft i and j. The model assumes that I_{ij} are independent for distinct aircraft pairs. If separation loss refers to a collision, then I_{ij} are *not* independent. For example, if there is a collision between aircraft 1 and 2 ($I_{12} = 1$), then there cannot be a collision between aircraft 2 and 3 ($I_{23} = 0$), assuming three-way collisions are practically impossible. However, if separation loss refers to a less severe event, like an NMAC or LOS, then the independence assumption is more reasonable. In these events, two aircraft pass by each other in close proximity. Their trajectories may be affected by the close encounter, but they otherwise continue flying. So, the probability of a separation loss with other aircraft is unaffected, assuming a random distribution of aircraft in space.[2]

Remark 3: To make the systems comparable, we assume that $p_{00} = p_0$ and $p_{11} = p_1$. In other words, with respect to a single aircraft pair, we assume that the failure of the function on the ground in the centralized system has the same effect as the failure of both functions in the air in the distributed system. Similarly, a working function on the ground in the centralized system has the same effect as two working functions in the air in the distributed system. A working-failed pair in the distributed system is assumed to yield a separation-loss probability p_{01} that is in between these two cases, $p_1 \leq p_{01} \leq p_0$.

III. Analysis of Basic Scenario

Let n be the number of aircraft in a region of airspace. The number of distinct aircraft pairs is $n(n-1)/2 = n^{(2)}/2$, where $n^{(k)} = n(n-1)\cdots(n-k+1)$ is the kth factorial moment of n. Let T_c be the number of separation losses occurring in the centralized system. Let T_d be the number of separation losses occurring in the distributed system. Implicitly, we assume the same number of time-coincident trajectory intersections between the two systems. We now give the expectation and variance of T_c and T_d and show that $\mathrm{var}[T_c] \geq \mathrm{var}[T_d]$ when $\mathrm{E}[T_c] = \mathrm{E}[T_d]$. All proofs are given in the appendix.

Theorem 1: The expected number of separation losses in the centralized system is

$$\mathrm{E}[T_c] = \frac{n^{(2)}}{2}[rp_0 + (1-r)p_1].$$

The expected number of separation losses in the distributed system is

$$\mathrm{E}[T_d] = \frac{n^{(2)}}{2}[q^2 p_0 + 2q(1-q)p_{01} + (1-q)^2 p_1].$$

This result is simply the the number of aircraft pairs $n^{(2)}/2$ multiplied by the separation-loss probability of one aircraft pair. This is a consequence of (1), in which the expectation can be brought inside the sum even when the terms are dependent. We now consider a special case when $p_{01} = p_1$.

Corollary 1: If $p_{01} = p_1$, then $\mathrm{E}[T_c] = \mathrm{E}[T_d]$ when $r = q^2$.

The condition $p_{01} = p_1$ implies that, for an aircraft pair in the distributed system, one working function is just as good as two working functions. This might be the case, for example, when only one aircraft needs to execute a resolution to avoid a separation loss.

To achieve equivalent levels of safety between the two systems, the corollary states that $r = q^2$. That is, the failure probability of the ground function must equal the square of the failure probability of the air-based function. (e.g., if $q = 10^{-5}$ then $r = 10^{-10}$). Intuitively, this is because two failures in the air is equivalent to one failure on the ground, when $p_{01} = p_1$. It may seem that this puts a much stricter requirement on the ground system. However, the failure probability of the ground function can be easily "squared" by adding dual-redundant equipment. From a cost perspective, this may be cheaper than providing analogous equipment on *every* airplane in the distributed system.

Theorem 2: The variance of the number of separation losses for the centralized system is

$$\begin{aligned}\mathrm{var}[T_c] &= \mathrm{E}[T_c] - \mathrm{E}^2[T_c] \\ &\quad + \frac{n^{(2)}}{2}\left(\frac{n^{(2)}}{2} - 1\right)[rp_0^2 + (1-r)p_1^2] \qquad (2)\end{aligned}$$

The variance of the number of separation losses for the distributed system is

$$\mathrm{var}[T_d] = \mathrm{E}[T_d] - \mathrm{E}^2[T_d] + \frac{(n-2)(n-3)}{n(n-1)}\mathrm{E}^2[T_d] + n^{(3)}C, \qquad (3)$$

where

$$\begin{aligned}C &= q^3 p_0^2 + 2q^2(1-q)p_0 p_{01} + q(1-q)p_{01}^2 \\ &\quad + 2q(1-q)^2 p_{01} p_1 + (1-q)^3 p_1^2.\end{aligned}$$

[2] The independence assumption is not perfect. For example, if there is an NMAC between aircraft 1 and 2, then there is heightened probability of a collision between aircraft 1 and 2, which potentially lowers the probability of an NMAC between aircraft 2 and 3. However, this contribution is small. The probability of a collision given that an NMAC has occurred can be approximated as the ratio of the collision volume divided by the NMAC volume. Using the proximity dimensions given, this ratio is $\pi 500^2 \cdot 100 / \pi 100^2 \cdot 30 \approx .01$.

This result cannot be obtained by multiplying a pairwise result by the number of aircraft pairs.

Theorem 3: If $\mathrm{E}[T_c] = \mathrm{E}[T_d]$ then $\mathrm{var}[T_c] \geq \mathrm{var}[T_d]$.

This result states that when the expectations of the two systems are equal, the variance of the centralized system is greater than or equal to the variance of the distributed system. The result is somewhat intuitive. If the ground function fails in the centralized system, it affects every aircraft, potentially resulting in multiple separation losses. If an air-based function fails in the distributed system, it affects only one aircraft (more precisely, it affects only pairs of aircraft involving the aircraft on which the function failed). Thus, the failure of the ground function is more extreme, resulting in a higher variance. However, we will show in Section V that this result does not necessarily extend to more complex models (Theorem 7).

IV. Numerical Evaluation of Model

This section gives a numerical comparison of $\mathrm{var}[T_c]$ and $\mathrm{var}[T_d]$ using rough order-of-magnitude estimates for model parameters. All else being equal, Theorem 3 favors the distributed system. But as we illustrate, the difference between $\mathrm{var}[T_c]$ and $\mathrm{var}[T_d]$ is negligible when the events of interest are rare. The difference is only noticeable when separation loss refers to something less extreme, like a 5-nm loss of separation.

To simplify the numerical examples in this section, we assume that $p_{01} = p_1$. Then Corollary 1 implies that $r = q^2$ when $\mathrm{E}[T_c] = \mathrm{E}[T_d]$. So the models can be specified by only three parameters: p_0, p_1 and r $(= q^2)$. Numerical estimates for these parameters depend on how separation loss is defined. Recall the definitions of p_0 and p_1 from Section II:

$$p_0 \equiv \Pr\{\text{separation loss} \mid \text{safety-related function failed}\},$$
$$p_1 \equiv \Pr\{\text{separation loss} \mid \text{safety-related function working}\}.$$

If the safety-related function is the ability to detect and resolve conflicts, then p_0 is the probability that a separation loss *would* occur in the absence of any conflict detection and resolution.

Numerical estimates for p_0 have been given in [17] for different definitions of separation loss. In that paper, the authors simulated aircraft trajectories in the U.S. using a 50% increase in operations above current levels. The model was run in a mode in which aircraft flew trajectories without trying to avoid other aircraft. Simulated collision events (in the absence of conflict detection and resolution) where roughly on the order of 10^{-5} to 10^{-4} per flight hour; near mid-air collisions were on the order of 10^{-4} to 10^{-3} per flight hour; 5-nm separation losses were on the order of 10^{-2} to 10^{-1} per flight hour. These numbers varied as a function of aircraft density within a sector and other factors such as whether or not the simulated flights followed great-circle routes or flight-plan routes. The objective here is simply to identify some reasonable approximate orders of magnitude for p_0.

Now p_1 is the probability of a separation loss when the conflict detection and resolution function is working. A proxy for p_1 might be the overall probability of occurrences of these

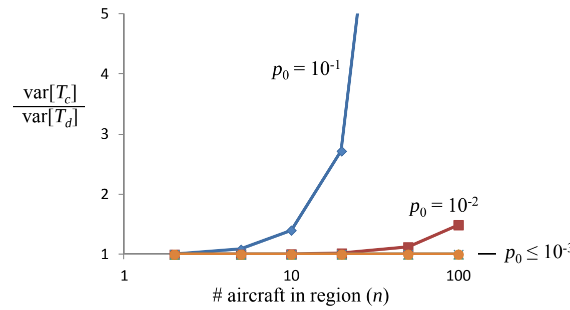

Fig. 2. Ratio of $\mathrm{var}[T_c]$ to $\mathrm{var}[T_d]$ with $p_1 = 10^{-6}$ and $r = 10^{-6}$.

events in today's system. Mid-air collision probabilities are on the order of 10^{-8} per flight hour [18]; near mid-air collisions might be on the order of 10^{-6} per flight hour;[3] 5-nm separation losses might be on the order of 10^{-5} per flight hour.[4]

In summary, the values of p_0 and p_1 are linked together via the definition of separation loss. For NMACs, reasonable values might be $p_0 \approx 10^{-4}$ and $p_1 \approx 10^{-6}$. For 5-nm separation losses, reasonable values might be $p_0 \approx 10^{-1}$ and $p_1 \approx 10^{-5}$. We assume that the ground-system failure probability is $r \approx 10^{-6}$. To compare $\mathrm{var}[T_c]$ and $\mathrm{var}[T_d]$, we use the following corollary:

Corollary 2: If $p_{01} = p_1$, then (3) simplifies to:

$$\mathrm{var}[T_d] = \mathrm{E}[T_d] - \frac{2}{n^{(2)}}\mathrm{E}^2[T_d] + n^{(3)} q^3 (1-q)(p_0 - p_1)^2. \quad (4)$$

Figure 2 shows sample ratios of $\mathrm{var}[T_c]/\mathrm{var}[T_d]$, computed using (2) and (4). In the figure, $p_1 = 10^{-6}$ and $r = 10^{-6}$. The x-axis is the number of aircraft n in the region. The ratio $\mathrm{var}[T_c]/\mathrm{var}[T_d]$ on the y-axis is always greater than or equal to 1, by Theorem 3. The ratio is increasing in n, because a failure of the ground function in the centralized system becomes relatively worse when there are more aircraft in the region. That is, a ground failure affects $(n^2 - n)/2$ aircraft pairs, growing quadratically in n, whereas an air failure affects $(n-1)$ aircraft pairs, growing linearly in n.

In the figure, $p_1 = 10^{-6}$, consistent with a scenario in which separation loss refers to a near mid-air collision. Thus, a reasonable associated number for p_0 might be 10^{-3} or 10^{-4}. From the figure, the ratio $\mathrm{var}[T_c]/\mathrm{var}[T_d]$, when $p_0 = 10^{-3}$ or $p_0 = 10^{-4}$, is effectively equal to one. The ratio is only large in the figure when $p_0 = 0.1$. However, this particular combination of values ($p_0 = 0.1$ and $p_1 = 10^{-6}$) may be unrealistic, because these values do not correspond in a consistent manner to the same definition of separation loss.

Figure 3 shows a similar plot in which $p_1 = 10^{-5}$. This value of p_1 is more consistent with a scenario in which separation loss refers to a 5-nm loss of separation. A reasonable associated number for p_0 might be 10^{-1} or 10^{-2}. The figure

[3]From Table 2-15 in [19], the number of pilot-reported NMACs in 2011 was 32; from Table 2-9, 17.8 million hours were flown; the ratio is on the order of 10^{-6} (part 121, commercial aviation).

[4]From [20], operational error rates are on the order of 2 per 10^5 operations (Figure 15, for tower and TRACON), which would translate to about 1 per 10^5 flight hours, assuming 2 flight hours per operation.

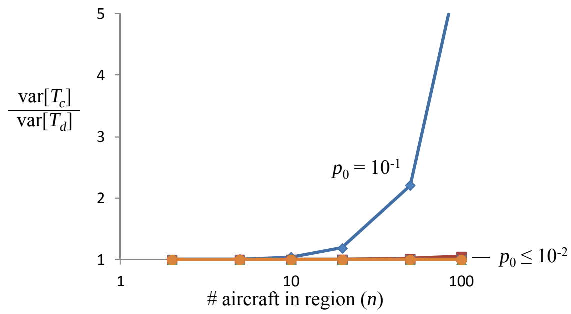

Fig. 3. Ratio of var[T_c] to var[T_d] with $p_1 = 10^{-5}$ and $r = 10^{-6}$.

shows that it is possible to have large ratios var[T_c]/var[T_d], at least when $p_0 = 10^{-1}$.

In summary, this section provided a numerical comparison of var[T_c]/var[T_d] using rough order-of-magnitude estimates for the parameter values. The parameters p_0 and p_1 are tied together via the definition of separation loss. For near mid-air collisions, the model suggests that var[T_c]/var[T_d] is very close to 1 (Figure 2). For less rare events, such as 5-nm separation losses, the model suggests that var[T_c]/var[T_d] could be much greater, depending on the specific parameter values (Figure 3). A higher ratio implies a higher probability of multiple simultaneous events for the centralized system, for a fixed expectation. Thus, while a failure of the ground function might result in multiple simultaneous 5-nm separation losses, it is very unlikely to result in multiple simultaneous NMACs.

V. Models for Multiple Safety-Related Functions

This section extends the analysis of the basic scenario to a more complex scenario in which there are multiple functions on the ground and in the air, as in Figure 4. This is more representative of real systems. In this scenario, both the centralized and distributed systems rely on multiple functions, some of which are implemented in the air and some of which are implemented on the ground. The main difference is that the centralized system has more ground-based functions, while the distributed system has more air-based functions.

To organize the model, the safety-related functions are grouped into the following subsets: (A) Functions that are implemented on the ground in the centralized system and in the air in the distributed system, (B) functions that are implemented on the ground in both systems, and (C) functions that are implemented in the air in both systems. We assume that both systems include the same sets of functions. That is, the two systems are identical except for the implementation of functions in subset (A). A comparative analysis of the two systems can be simplified by the following two observations:

Observation 1: Each subset of functions can be abstracted into a single function that takes multiple states. For example, if subset (A) contains m functions that are either working or failed, the state of these functions can be described by a single function taking one of 2^m different states.

Observation 2: For the purpose of comparing the two systems, it suffices to analyze the systems with respect to the functions in subset (A), ignoring the common effects of functions in subsets (B) and (C). A formal argument for this observation is given in the Appendix.

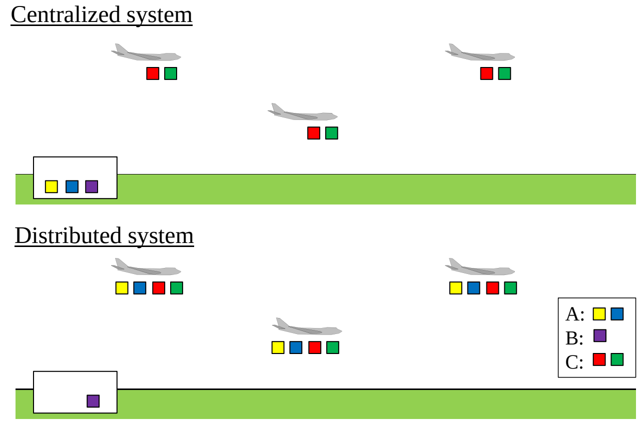

Fig. 4. Comparison of a centralized and distributed system.

Based on these two observations, we propose the following reliability model to compare the centralized and distributed systems:

Centralized system: Let the "ground system" denote the set of functions in subset (A). Let A_c denote the random state of the ground system, where $A_c \in \{0, 1, \ldots, s\}$. Let $r_j = \Pr\{A_c = j\}$. For each distinct aircraft pair, the probability of a separation loss depends on A_c. If $A_c = j$, the separation-loss probability is p_j. Without loss of generality, $A_c = s$ represents the safest state (e.g., all ground functions are working), $A_c = 0$ represents the most dangerous state (e.g., all ground functions have failed), and the intermediate states are ordered with respect to safety: $p_s \leq \cdots \leq p_1 \leq p_0$. Given A_c, separation losses between distinct aircraft pairs are independent.

Distributed system: Let the "air system" denote the set of functions in subset (A) on each aircraft. Let A_d^i denote the random state of the air system on aircraft i, where $A_d^i \in \{0, 1, \ldots, s\}$. Let $q_j = \Pr\{A_d^i = j\}$. The states A_d^i are assumed to be independent of each other. For each distinct aircraft pair (i, j), the probability of a separation loss depends on A_d^i and A_d^j. Given that $A_d^i = k$ and $A_d^j = l$, the separation-loss probability is p_{kl}. Conditional on A_d^i, separation losses between distinct aircraft pairs are independent.

To make the systems comparable, we assume that $p_{kk} = p_k$. In other words, with respect to a single aircraft pair, if two aircraft are in state k in the distributed system, then the separation-loss probability is the same as in the centralized system when the ground system is in state k. It is also assumed that $p_k \leq p_{kl} \leq p_l$ when $k \geq l$. In other words, in the distributed system, when two aircraft are in different states, the separation-loss probability is at least as large as the probability when both aircraft are in the safer state and no larger than the probability when both aircraft are in the more dangerous state.

We now give the expectation and variance of T_c and T_d. All proofs are given in the appendix.

Theorem 4: The expected number of separation losses in the centralized system is

$$\mathrm{E}[T_c] = \frac{n^{(2)}}{2} \sum_i p_i r_i.$$

The expected number of separation losses in the distributed system is

$$\mathrm{E}[T_d] = \frac{n^{(2)}}{2} \sum_{i,j} p_{ij} q_i q_j.$$

Theorem 5: The variance of the number of separation losses in the centralized system is

$$\mathrm{var}[T_c] = \mathrm{E}[T_c] - \mathrm{E}^2[T_c] + \frac{n^{(2)}}{2}\left(\frac{n^{(2)}}{2} - 1\right)\sum_i r_i p_i^2. \quad (5)$$

The variance of the number of separation losses in the distributed system is

$$\mathrm{var}[T_d] = \mathrm{E}[T_d] - \mathrm{E}^2[T_d] + \frac{(n-2)(n-3)}{n(n-1)}\mathrm{E}^2[T_d] + n^{(3)} \sum_{i,j,k} q_i q_j q_k p_{ij} p_{ik}. \quad (6)$$

The expectation in Theorem 4 is simply the number of aircraft pairs $n^{(2)}/2$ multiplied by the pairwise separation-loss probability. The variance in Theorem 5, however, cannot be obtained from a pairwise model. In the basic scenario, we had $\mathrm{var}[T_c] \geq \mathrm{var}[T_d]$ when $\mathrm{E}[T_c] = \mathrm{E}[T_d]$ (Theorem 3). In this general scenario, we have a slightly weaker result:

Theorem 6: If $\mathrm{E}[T_c] = \mathrm{E}[T_d]$, then there exists an n^* such that $n \geq n^*$ implies that $\mathrm{var}[T_c] > \mathrm{var}[T_d]$ (provided there is no i such that $p_i = 1$).

The result is somewhat intuitive. As the number of aircraft in the region increases, ground failures correspond to more extreme events, increasing the variance relative to the distributed system. However, we can show via counter-example that there are cases where $\mathrm{var}[T_c] < \mathrm{var}[T_d]$ even when $\mathrm{E}[T_c] = \mathrm{E}[T_d]$.

Theorem 7: When $s > 2$ and $\mathrm{E}[T_c] = \mathrm{E}[T_d]$, it is not necessarily true that $\mathrm{var}[T_c] \geq \mathrm{var}[T_d]$.

The case $s = 2$ corresponds to the basic scenario (one function with two states), which was covered by Theorem 3. The case $s > 2$ corresponds to the general scenario in which there are multiple functions in subset (A), as in Figure 4.

To prove Theorem 7, we produce a counter-example. Let $s = 3$ (the safety-related function can take one of three states). Let $r_0 = .1$, $r_1 = .8$, $r_2 = .1$ (in the centralized system, the ground system is most likely to be in the middle state). Let $q_0 = .5$, $q_1 = 0$, $q_2 = .5$ (in the distributed system, the air system is most likely to be in one of the end states). Let $p_0 = 1$, $p_1 = .5$, and $p_2 = 0$ (state 0 is the most dangerous state, state 2 is the safest state); let $p_{02} = .5$; p_{01} and p_{12} are irrelevant since $q_1 = 0$. From Theorem 4, it can be checked that $\mathrm{E}[T_c] = \mathrm{E}[T_d] = 0.5(n^{(2)}/2)$. From Theorem 5, it can be checked that

$$\begin{aligned}\mathrm{var}[T_c] - \mathrm{var}[T_d] &= \frac{n^{(2)}}{2}\left(\frac{n^{(2)}}{2} - 1\right)\sum_i r_i p_i^2 \\ &\quad - \frac{(n-2)(n-3)}{n(n-1)}\mathrm{E}^2[T_d] - n^{(3)} \sum_{i,j,k} q_i q_j q_k p_{ij} p_{ik} \\ &= \frac{n^{(2)}}{2}\left(\frac{n^{(2)}}{2} - 1\right).3 - \frac{n^{(4)}}{4}.25 - n^{(3)}.3125.\end{aligned}$$

If $n = 3$, then $\mathrm{var}[T_c] - \mathrm{var}[T_d] = 3(2)(.3) - 0(.25) - 6(.3125) < 0$, demonstrating the counter-example. As n gets large, the equation is dominated by terms involving n^4, so eventually $\mathrm{var}[T_c] > \mathrm{var}[T_d]$ (since $.3 > .25$), consistent with Theorem 6.

Intuitively, counter examples can be constructed by choosing parameters such that the centralized system has a high probability of being in middle states (reducing its variance), while the distributed system has a high probability of being in end states (increasing its variance). When $s > 2$, there are enough degrees of freedom to achieve this while simultaneously satisfying the condition that $\mathrm{E}[T_c] = \mathrm{E}[T_d]$. When $s = 2$, fewer degrees of freedom prevent the possibility of constructing such examples.

VI. Summary and Conclusions

This paper presented analytical models to compare centralized and distributed separation assurance systems. In a centralized system, safety-related functions are implemented by common equipment on the ground. In a distributed system, safety-related functions are implemented by equipment on each aircraft. Intuitively, failures on the ground are worse than failures in the air because ground failures affect multiple aircraft. The objective of this paper is to explore the validity of this intuition.

To do this, we developed safety models accounting for dependencies associated with failures affecting more than two aircraft at a time (a feature not possible with models that consider pairs of aircraft in isolation). From the models, we derived analytic formulas for the expectation and variance of the number of separation losses. This was done for a basic scenario involving a single component/function as well as for an extended scenario involving multiple components/functions on the ground and in the air.

In the basic scenario, we found that the variance of the number of separation losses was always higher for the centralized system, holding the expectations equal. This indicates a greater possibility for multiple separation losses occurring simultaneously due to the failure of the safety-related function. However, numerical examples showed that this difference was negligible when the events of interest were rare. Furthermore, in the extended scenario, it was possible for the variance associated with the distributed system to be higher. The overall implication is that the common-cause failure of the ground function (an issue that affects the centralized system, but not the distributed system) may not have such a critical role in differentiating the two systems. A full comparison of centralized and distributed systems must also consider other factors such as situational awareness, response times, coordinated versus uncoordinated trajectories and so forth.

One potential application of the framework in this paper is to adapt it to more complex event-tree models, such as those in [8] and [10]. Those models address many additional complexities, such as the time-dependent nature of detecting separation losses and the associated response times. However, the models treat safety in a pairwise manner. Approximating those more complex models via the framework in this paper could provide a mechanism to perform a more precise region-wide analysis. Another potential application is unmanned aerial systems (UAS). For example, the models in this paper could be used to compare concepts in which unmanned aircraft are managed by a single pilot/controller on the ground versus concepts in which the aircraft are managed in a more distributed or autonomous way.

VII. Appendix: Theorem Proofs

Proof of Theorems 1 and 2: Theorem 1 is a special case of Theorem 4; Theorem 2 is a special case of Theorem 5: Replace r_0 with r, r_1 with $1-r$, q_0 with q, and q_1 with $1-q$. □

Proof of Corollary 1: Setting $p_{01} = p_1$ in Theorem 1 and comparing $\mathrm{E}[T_d] = \mathrm{E}[T_c]$ gives the result. □

Proof of Corollary 2: Rewrite (3) as

$$\mathrm{var}[T_d] = \mathrm{E}[T_d] - \mathrm{E}^2[T_d]\left(\frac{2}{n^{(2)}} + \frac{4n^{(3)}}{n^{(2)}n^{(2)}}\right) + n^{(3)}C, \quad (7)$$

where we have used

$$\begin{aligned} 1 - \frac{(n-2)(n-3)}{n(n-1)} &= \frac{4n-6}{n(n-1)} = \frac{2+4(n-2)}{n^{(2)}} \\ &= \frac{2}{n^{(2)}} + \frac{4n^{(3)}}{n^{(2)}n^{(2)}}. \end{aligned}$$

Now, when $p_{01} = p_1$, $\mathrm{E}[T_d] = (n^{(2)}/2)[q^2 p_0 + (1-q^2)p_1]$ and

$$\begin{aligned} C &= q^3 p_0^2 + 2q^2(1-q)p_0p_1 \\ &\quad + [q(1-q) + 2q(1-q)^2 + (1-q)^3]p_1^2 \\ &= q^3p_0^2 + 2q^2(1-q)p_0p_1 + [1-2q^2+q^3]p_1^2. \end{aligned}$$

Then

$$\begin{aligned} -\mathrm{E}^2[T_d]\frac{4n^{(3)}}{n^{(2)}n^{(2)}} + n^{(3)}C &= -n^{(3)}[q^2p_0 + (1-q^2)p_1]^2 \\ &\quad + n^{(3)}[q^3p_0^2 + 2q^2(1-q)p_0p_1 + (1-2q^2+q^3)p_1^2 \\ &= n^{(3)}[(q^3-q^4)p_0^2 + (2q^2(1-q) - 2q^2(1-q^2))p_0p_1 \\ &\qquad + (q^3-q^4)p_1^2] \\ &= n^{(3)}(q^3-q^4)(p_0-p_1)^2. \end{aligned}$$

Substituting this into (7) gives the result. □

Proof of Theorem 3: Subtract (3) from (2):

$$\begin{aligned} \mathrm{var}[T_c] - \mathrm{var}[T_d] &= \frac{n^{(2)}}{2}\left(\frac{n^{(2)}}{2} - 1\right)[rp_0^2 + (1-r)p_1^2] \\ &\quad - \frac{(n-2)(n-3)}{n(n-1)}\mathrm{E}^2[T_d] - n^{(3)}C. \quad (8) \end{aligned}$$

Let $A \equiv rp_0^2 + (1-r)p_1^2$ and $B \equiv [rp_0 + (1-r)p_1]^2$. From Theorem 1, $\mathrm{E}^2[T_d] = Bn^{(2)}n^{(2)}/4$ (assuming $\mathrm{E}[T_d] = \mathrm{E}[T_c]$). Also, $(n^{(2)}/2) - 1 = (n^2 - n - 2)/2 = (n-2)(n+1)/2$. So (8) can be written

$$\begin{aligned} \mathrm{var}[T_c] - \mathrm{var}[T_d] &= \frac{n^{(3)}(n+1)}{4}A - \frac{n^{(4)}}{4}B - n^{(3)}C \\ &= \frac{n^{(3)}}{4}[(n+1)A - (n-3)B - 4C] \\ &= \frac{n^{(3)}}{4}[(n-1)(A-B) + 2(A+B-2C)]. \quad (9) \end{aligned}$$

$A - B$ can be simplified as follows:

$$\begin{aligned} A - B &= rp_0^2 + (1-r)p_1^2 - [rp_0 + (1-r)p_1]^2 \\ &= (r - r^2)p_0^2 + [(1-r) - (1-r)^2]p_1^2 - 2r(1-r)p_0p_1 \\ &= r(1-r)(p_0 - p_1)^2 \ge 0. \quad (10) \end{aligned}$$

Thus, to prove that (9) ≥ 0, it remains to show that $A + B - 2C \ge 0$. Let $p_{01} \equiv ap_0 + (1-a)p_1$, where a $(0 \le a \le 1)$ measures the relative position of p_{01} within the range $[p_1, p_0]$. Letting $\bar{q} \equiv 1-q$ and $\bar{a} \equiv 1-a$, the assumption $\mathrm{E}[T_c] = \mathrm{E}[T_d]$ implies

$$\begin{aligned} rp_0 + (1-r)p_1 &= q^2p_0 + 2q\bar{q}p_{01} + \bar{q}^2p_1. \\ rp_0 + (1-r)p_1 &= q^2p_0 + 2q\bar{q}[ap_0 + \bar{a}p_1] + (q^2 - 2q + 1)p_1 \\ r(p_0 - p_1) &= (q^2 + 2q\bar{q}a)p_0 + (2q\bar{q}\bar{a} + q^2 - 2q)p_1 \\ r &= q^2 + 2q\bar{q}a. \quad (11) \end{aligned}$$

Substituting (11) into (10) gives

$$\begin{aligned} A - B &= (q^2 + 2q\bar{q}a)(1 - q^2 - 2q\bar{q}a)(p_0 - p_1)^2 \\ &= q(q + 2\bar{q}a)\bar{q}(1 + q - 2qa)(p_0 - p_1)^2 \\ &= q\bar{q}(q + q^2 - 2q^2a + 2\bar{q}a + 2q\bar{q}a - 4q\bar{q}a^2)(p_0 - p_1)^2 \\ &= q\bar{q}(q + q^2 + 2a - 4qa^2 - 4q^2a\bar{a})(p_0 - p_1)^2. \quad (12) \end{aligned}$$

Now we derive an expression for $A - C$. Substituting (11) into the expressions for A and C:

$$\begin{aligned} A - C &= (q^2 + 2q\bar{q}a - q^3)p_0^2 - 2q^2\bar{q}p_0(ap_0 + \bar{a}p_1) \\ &\quad - q\bar{q}(ap_0 + \bar{a}p_1)^2 - 2q\bar{q}^2(ap_0 + \bar{a}p_1)p_1 \\ &\quad + (1 - q^2 - 2q\bar{q}a - \bar{q}^3)p_1^2. \end{aligned}$$

Group this into terms involving p_0^2, p_0p_1 and p_1^2. The terms involving p_0^2 are:

$$\begin{aligned} &(q^2 - q^3 + 2q\bar{q}a - 2q^2\bar{q}a - q\bar{q}a^2)p_0^2 \\ &= q\bar{q}(q + 2a - 2qa - a^2)p_0^2 \\ &= q\bar{q}(q\bar{a} + \bar{q}a + a\bar{a})p_0^2. \quad (13) \end{aligned}$$

The terms involving p_0p_1 are:

$$\begin{aligned} &(-2q^2\bar{q}\bar{a} - 2q\bar{q}^2a - 2q\bar{q}a\bar{a})p_0p_1 \\ &\qquad = -2q\bar{q}(q\bar{a} + \bar{q}a + a\bar{a})p_0p_1. \quad (14) \end{aligned}$$

The terms involving p_1^2 are:

$$\begin{aligned}&(-q\bar{q}\bar{a}^2 - 2q\bar{q}^2\bar{a} - 2aq\bar{q} + 1 - q^2 - \bar{q}^3)p_1^2 \\ &= q\bar{q}(-\bar{a}^2 - 2\bar{q}\bar{a} - 2a + 3 - q)p_1^2 \\ &= q\bar{q}(-1 + 2a - a^2 - 2 + 2a - 2q + aq - 2a + 3 - q)p_1^2 \\ &= q\bar{q}(q\bar{a} + \bar{q}a + a\bar{a})p_1^2. \qquad (15)\end{aligned}$$

Combining (13), (14) and (15) gives

$$A - C = (p_0 - p_1)^2 q\bar{q}[q\bar{a} + \bar{q}a + a\bar{a}]. \qquad (16)$$

To finish the proof, we show that $A + B - 2C = 2(A - C) - (A - B) \geq 0$, using (16) and (12):

$$\begin{aligned}&\frac{2 \cdot (16) - (12)}{q\bar{q}(p_0 - p_1)^2} = 2(q\bar{a} + \bar{q}a + a\bar{a}) \\ &\qquad - (q + q^2 + 2a - 4qa^2 - 4q^2a\bar{a}) \\ &= 2q - 4qa + 2a + 2a\bar{a} - q - q^2 - 2a + 4qa^2 + 4q^2a\bar{a} \\ &= q - q^2 - 4qa + 2a\bar{a} + 4qa^2 + 4q^2a\bar{a} \\ &= q\bar{q} + 2a\bar{a}(1 - 2q + q^2) = q\bar{q} + 2a\bar{a}(q + \bar{q}^2) \geq 0. \qquad \square\end{aligned}$$

Proof of Theorem 4: For the centralized system, condition on the ground state A_c. $\mathrm{E}[T_c|A_c] = (n^{(2)}/2)p_{A_c}$, where $A_c = i$ with probability r_i. Thus,

$$\mathrm{E}[T_c] = \mathrm{E}[\mathrm{E}[T_c|A_c]] = \mathrm{E}\left[\frac{n^{(2)}}{2}p_{A_c}\right] = \frac{n^{(2)}}{2}\sum_i p_i r_i.$$

Similarly, for the distributed system, condition on the states of the airborne functions $A_d^1, \ldots, A_d^n$:

$$\begin{aligned}\mathrm{E}[T_d] &= \mathrm{E}[\mathrm{E}[T_d|A_d^1, \ldots, A_d^n]] = \mathrm{E}\left[\mathrm{E}\left[\sum_{i<j} I_d^{ij}|A_d^1, \ldots, A_d^n\right]\right] \\ &= \mathrm{E}\left[\sum_{i<j}\mathrm{E}[I_d^{ij}|A_d^1, \ldots A_d^n]\right] = \mathrm{E}\left[\sum_{i<j} p_{A_d^i A_d^j}\right] \\ &= \frac{n^{(2)}}{2}\mathrm{E}[p_{A_d^i, A_d^j}] = \frac{n^{(2)}}{2}\sum_{k,l} p_{kl}q_k q_l. \qquad \square\end{aligned}$$

Proof of Theorem 5: For the centralized system, condition on the ground state A_c. Given A_c, the number of separation losses is a binomial random variable with parameters $m \equiv n^{(2)}/2$ and p_{A_c}. So,

$$\begin{aligned}\mathrm{E}[T_c^2] &= \mathrm{E}[\mathrm{E}[T_c^2|A_c]] = \mathrm{E}[(m^2 - m)p_{A_c}^2 + mp_{A_c}] \\ &= \sum_i [(m^2 - m)p_i^2 + mp_i]r_i \\ &= \mathrm{E}[T_c] + (m^2 - m)\sum_i r_i p_i^2.\end{aligned}$$

Applying $\mathrm{var}[T_c] = \mathrm{E}[T_c^2] - \mathrm{E}^2[T_c]$ gives (5).

For the distributed system, condition on the airborne states A_d^i for each aircraft i. Let N_i be the number of aircraft in state i (that is, the number of indices j such that $A_d^j = i$).

$$\begin{aligned}\mathrm{E}[T_d^2] &= \mathrm{E}[\mathrm{E}[T_d^2|N_1, \ldots, N_s]] \\ &= \mathrm{E}\left[\mathrm{var}[T_d|N_1, \ldots, N_s] + \mathrm{E}^2[T_d|N_1, \ldots, N_s]\right]. \qquad (17)\end{aligned}$$

$N_i(N_i - 1)/2$ is the number of distinct aircraft pairs such that both aircraft are in state i. N_iN_j is the number of distinct aircraft pairs such that one aircraft is in state i and the other is in state j ($i \neq j$). Thus, given $N_1, \ldots, N_s$, the number of separation losses T_d is a sum of independent binomial random variables:

$$\begin{aligned}\{T_d|N_1, \ldots, N_s\} \sim \sum_i \mathrm{bin}(N_i(N_i - 1)/2, p_i) \\ + \sum_{i<j}\mathrm{bin}(N_iN_j, p_{ij}). \qquad (18)\end{aligned}$$

Then,

$$\begin{aligned}\mathrm{var}[T_d|N_1, \ldots, N_s]] = \sum_i \frac{N_i(N_i - 1)}{2}p_i(1 - p_i) \\ + \sum_{i<j} N_iN_jp_{ij}(1 - p_{ij}).\end{aligned}$$

Now, $\{N_1, \ldots, N_n\}$ follow a multinomial distribution with parameters n and $q_1, \ldots, q_n$. Using properties of the multinomial distribution and the symmetry of p_{ij} $(= p_{ji})$,

$$\begin{aligned}&\mathrm{E}[\mathrm{var}[T_d|N_1, \ldots, N_s]] \\ &= \frac{1}{2}\sum_i \mathrm{E}[N_i^{(2)}]p_i(1 - p_i) + \frac{1}{2}\sum_{i\neq j}\mathrm{E}[N_iN_j]p_{ij}(1 - p_{ij}) \\ &= \frac{1}{2}\sum_i n^{(2)}q_i^2p_i(1 - p_i) + \frac{1}{2}\sum_{i\neq j} n^{(2)}q_iq_jp_{ij}(1 - p_{ij}) \\ &= \frac{n^{(2)}}{2}\sum_{i,j} q_iq_jp_{ij}(1 - p_{ij}) \\ &= \mathrm{E}[T_d] - \frac{n^{(2)}}{2}\sum_{i,j} q_iq_jp_{ij}^2. \qquad (19)\end{aligned}$$

Similarly, from (18) and the symmetry of p_{ij}:

$$\begin{aligned}\mathrm{E}[T_d|N_1, \ldots N_s] &= \frac{1}{2}\sum_i (N_i^2 - N_i)p_i + \frac{1}{2}\sum_{i\neq j} N_iN_jp_{ij} \\ &= \frac{1}{2}\sum_{i,j} N_iN_jp_{ij} - \frac{1}{2}\sum_i N_ip_i.\end{aligned}$$

Squaring this result and taking the expectation gives:

$$\begin{aligned}\mathrm{E}[\mathrm{E}^2[T_d|N_1, \ldots N_s]] = \frac{1}{4}\sum_{i,j,k,l}\mathrm{E}[N_iN_jN_kN_lp_{ij}p_{kl}] \\ - \frac{1}{2}\sum_{i,j,k}\mathrm{E}[N_iN_jN_kp_{ij}p_k] + \frac{1}{4}\sum_{i,j}\mathrm{E}[N_iN_jp_ip_j]. \qquad (20)\end{aligned}$$

Now we substitute moments of the multinomial distribution (e.g., [21], [22]). Evaluating (20) term by term (ignoring the leading constants for the moment):

$$\begin{aligned}\sum_{i,j}\mathrm{E}[N_iN_jp_ip_j] &= \sum_{i\neq j}\mathrm{E}[N_iN_j]p_ip_j + \sum_i \mathrm{E}[N_i^2]p_i^2 \\ &= \sum_{i\neq j} n^{(2)}q_iq_jp_ip_j + \sum_i (n^{(2)}q_i^2 + nq_i)p_i^2 \\ &= n^{(2)}\sum_{i,j} q_iq_jp_ip_j + n\sum_i q_ip_i^2. \qquad (21)\end{aligned}$$

To keep track of subscripts, let $f_{ijk} \equiv p_{ij}p_k$. Then

$$\begin{aligned}
&\sum_{i,j,k} \mathrm{E}[N_iN_jN_kp_{ij}p_k] = \sum_{i\neq j\neq k} \mathrm{E}[N_iN_jN_k]f_{ijk} \\
&\quad + \sum_{i\neq j} \mathrm{E}[N_i^2N_j](f_{iij}+f_{iji}+f_{jii}) + \sum_i \mathrm{E}[N_i^3]f_{iii} \\
&= \sum_{i\neq j\neq k} n^{(3)}q_iq_jq_kf_{ijk} \\
&\quad + \sum_{i\neq j}(n^{(3)}q_i^2q_j + n^{(2)}q_iq_j)(f_{iij}+f_{iji}+f_{jii}) \\
&\quad + \sum_i (n^{(3)}q_i^3 + 3n^{(2)}q_i^2 + nq_i)f_{iii} \\
&= n^{(3)}\sum_{i,j,k} q_iq_jq_kf_{ijk} + n^{(2)}\sum_{i,j} q_iq_j(f_{iij}+f_{iji}+f_{jii}) \\
&\quad + n\sum_i q_if_{iii} \\
&= n^{(3)}\sum_{i,j,k} q_iq_jq_kp_{ij}p_k + n^{(2)}\sum_{i,j} q_iq_j(p_ip_j+2p_ip_{ij}) \\
&\quad + n\sum_i q_ip_i^2 \qquad (22)
\end{aligned}$$

Similarly, it can be shown (the proof is omitted):

$$\begin{aligned}
&\sum_{i,j,k,l} \mathrm{E}[N_iN_jN_kN_lp_{ij}p_{kl}] = n^{(4)}\sum_{i,j,k,l} q_iq_jq_kq_lp_{ij}p_{kl} \\
&+ n^{(3)}\sum_{i,j,k} q_iq_jq_k(2p_{ij}p_k + 4p_{ij}p_{ik}) \\
&+ n^{(2)}\sum_{i,j} q_iq_j(p_ip_j + 4p_ip_{ij} + 2p_{ij}^2) + n\sum_i q_ip_i^2. \qquad (23)
\end{aligned}$$

Substituting (21), (22), and (23) into (20) gives:

$$\begin{aligned}
\mathrm{E}[\mathrm{E}^2[T_d|N_1,\dots N_s]] &= \frac{n^{(4)}}{4}\sum_{i,j,k,l} q_iq_jq_kq_lp_{ij}p_{kl} \\
&+ n^{(3)}\sum_{i,j,k,} q_iq_jq_kp_{ij}p_{ik} + \frac{n^{(2)}}{2}\sum_{i,j} q_iq_jp_{ij}^2. \qquad (24)
\end{aligned}$$

Substituting (24) and (19) into (17) and using

$$\frac{n^{(4)}}{4}\sum_{i,j,k,l} q_iq_jq_kq_lp_{ij}p_{kl} = \frac{(n-2)(n-3)}{n(n-1)}\mathrm{E}^2[T_c] \qquad (25)$$

(derived from Theorem 4) gives the result in (6). □

Proof of Theorem 6: Subtracting (6) from (5) (assuming $\mathrm{E}[T_c] = \mathrm{E}[T_d]$) and using (25) gives:

$$\begin{aligned}
\mathrm{var}[T_c] - \mathrm{var}[T_d] &= \frac{n^{(2)}}{2}\left(\frac{n^{(2)}}{2} - 1\right)\sum_i r_ip_i^2 \\
&- \frac{n^{(4)}}{4}\sum_{i,j,k,l} q_iq_jq_kq_lp_{ij}p_{kl} - n^{(3)}\sum_{i,j,k} q_iq_jq_kp_{ij}p_{ik}.
\end{aligned}$$

To show that $\mathrm{var}[T_c] - \mathrm{var}[T_d] > 0$ for arbitrarily large n, it suffices to compare terms associated with the highest power of n (namely n^4). In other words, the theorem is proved by showing that:

$$\sum_i r_ip_i^2 > \sum_{i,j,k,l} q_iq_jq_kq_lp_{ij}p_{kl}.$$

This is shown as follows:

$$\begin{aligned}
\sum_i r_ip_i^2 &> \left(\sum_i r_ip_i\right)^2 = \left(\sum_{i,j} q_iq_jp_{ij}\right)^2 \\
&= \sum_{i,j,k,l} q_iq_jq_kq_lp_{ij}p_{kl}.
\end{aligned}$$

The inequality follows from $\mathrm{E}[X^2] > \mathrm{E}^2[X]$, where X is a random variable taking the value p_i with probability r_i, provided there is no i such that $p_i = 1$. The first equality follows from $\mathrm{E}[T_c] = \mathrm{E}[T_d]$. □

Discussion of Observation 2: The observation that the common effects of functions in subsets (B) and (C) can be ignored is somewhat intuitive. This section lays out the precise conditions under which this observation can be made.

Define the following random variables associated with the *centralized* system: Let A_c be the state of functions in subset (A); let B_c be the state of functions in subset (B); let C_c^i be the state of functions in subset (C) for aircraft i; let $I_c^{ij} = 1$ if there is a separation loss between aircraft i and j, 0 otherwise. Similarly, define the following random variables associated with the *distributed* system: Let A_d^i be the state of functions in subset (A) for aircraft i; let B_d be the state of functions in subset (B); let C_d^i be the state of functions in subset (C) for aircraft i; let $I_d^{ij} = 1$ if there is a separation loss between aircraft i and j, 0 otherwise.

In order to "derive" Observation 2, as well as related modeling assumptions given in Section V, we make the following more fundamental assumptions: Suppose that A_c, B_c, C_c^i, A_d^i, B_d, and C_d^i are mutually independent. In addition, assume:

$$\Pr\{B_c = b\} = \Pr\{B_d = b\},\ \Pr\{C_c^i = c\} = \Pr\{C_d^i = c\}, \qquad (26)$$

$$\begin{aligned}
&\Pr\{I_c^{ij} = 1|A_c = a, B_c = b, C_c^i = c_1, C_c^j = c_2\} = \\
&\Pr\{I_d^{ij} = 1|A_d^i = A_d^j = a, B_d = b, C_d^i = c_1, C_d^j = c_2\}, \qquad (27)
\end{aligned}$$

and, when $k > l$,

$$\begin{aligned}
&\Pr\{I_d^{ij} = 1|A_d^i = k, A_d^j = k, B_d = b, C_d^i = c_1, C_d^j = c_2\} \\
\le&\Pr\{I_d^{ij} = 1|A_d^i = k, A_d^j = l, B_d = b, C_d^i = c_1, C_d^j = c_2\} \\
\le&\Pr\{I_d^{ij} = 1|A_d^i = l, A_d^j = l, B_d = b, C_d^i = c_1, C_d^j = c_2\}, \qquad (28)
\end{aligned}$$

for all i and j, where a, b, and c_1 (and c_2) are possible states of functions in subsets (A), (B), and (C), respectively. Assumption (26) states that the two systems have equivalent state probabilities with respect to the common functions in (B) and (C). Assumption (27) states that, all else being equal, the separation-loss probabilities of the two systems are the same when the functions in (A) are in the same state ($A_c = A_d^i = A_d^j$). Equation (29) states that, all else being equal, when two aircraft are in different states in the distributed system ($A_d^i = k$ and $A_d^j = l$), the separation-loss probability is in between the case when both aircraft are in the safer state and the case when both aircraft are in the more dangerous state.

In Section V, we defined the following probabilities: $p_k \equiv \Pr\{I_c^{ij} = 1 | A_c = k\}$ and $p_{kl} \equiv \Pr\{I_d^{ij} = 1 | A_d^i = k, A_d^j = l\}$. The previous assumptions imply:

$$p_k = p_{kk}, \quad (29)$$

$$p_k \leq p_{kl} \leq p_l \text{ when } k > l. \quad (30)$$

Both results can be proved by applying the law of total probability. For example, to prove (29):

$$\begin{aligned} p_k &= \Pr\{I_c^{ij} = 1 | A_c = k\} \\ &= \sum_{b, c_1, c_2} \Pr\{I_c^{ij} = 1 | A_c = k, B_c = b, C_c^i = c_1, C_c^j = c_2\} \\ &\qquad \times \Pr\{B_c = b\} \Pr\{C_c^i = c_1\} \Pr\{C_c^j = c_2\} \\ &= \sum_{b, c_1, c_2} \Pr\{I_d^{ij} = 1 | A_d^i = A_d^j = k, B_d = b, C_d^i = c_1, C_d^j = c_2\} \\ &\qquad \times \Pr\{B_d = b\} \Pr\{C_d^i = c_1\} \Pr\{C_d^j = c_2\} \\ &= \Pr\{I_d^{ij} = 1 | A_d^i = k, A_d^j = k\} = p_{kk}. \end{aligned}$$

The second equation is derived in a similar manner. In summary, this discussion has specified fundamental assumptions regarding functions in subsets (A), (B), and (C). These assumptions imply that the model can be stated purely in terms of functions in subset (A) via the probabilities p_k and p_{kl}. Further, these assumptions imply (29) and (30), which are modeling assumptions stated in Section V. □

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